Grade 4

Number Talks

Purposes:

To develop computational fluency (accuracy, efficiency, flexibility) in order to focus students' attention so they will move from:

- figuring out the answers any way they can to . . .
- becoming more efficient at figuring out answers to . . .
- just knowing or using efficient strategies

Description:

The teacher gives the class an equation to solve mentally. Students may use pencil and paper to keep track of the steps as they do the mental calculations. Students' strategies are shared and discussed to help all students think more flexibly as they work with numbers and operations.

Materials:

- Prepared problems to be explored
- Chalkboard, white board, or overhead transparency
- Individual white boards or pencil and paper
- Optional: Interlocking cubes and/or base ten materials

Time: 15 minutes maximum

Directions:

Example: 16 x 25

1. Write an expression **horizontally** on the board (e.g., 16 x 25).
2. Ask students to think first and estimate their answer before attempting to solve the problem. Post estimates on the board. This will allow you to see how the students are developing their number sense and operational sense.
3. Ask students to mentally find the solution using a strategy that makes sense to them. Encourage students to "think first" and then check with models, if needed. Have tools available to help students visualize the problem if they need them (e.g., interlocking cubes, base ten blocks).

4. Ask students to explain to a partner how they solved the problem.

5. While students are discussing their strategies, walk among the groups listening to the explanations. Find those strategies you want to call attention to for the whole class. Choose strategies for discussion that you might want other students to think about and possibly experiment with.

6. Call on a student to fully explain the steps he/she followed to solve the problem.

7. Record the steps precisely as the student explains them to you. Ask clarifying questions as needed to ensure that you understand the flow of the child's thinking. Be explicit about the mathematics.

   - "You said you broke up the 16 into 4 x 4. Why did you do that?"
   - "Does this strategy always work? How do you know?"
   - "What did you know about the number 16 that allowed you to do that?"

8. As time allows, ask other students to share different methods they used for solving the equation. Follow up on each strategy shared by asking similar questions to those included in step 7. Publicly record these methods as well.

9. It is very important to facilitate a discussion about how the different representations/strategies relate to each other and result in the same answer.

   See the following examples:

Example - Guiding the Share-Out:

Scenario 1: 16 x 25

Public Recording	Student's Explanation
16 = 4 x 4 4 x 25 = 100 4 x 100 = 400	"I know that 16 is the same as 4 x 4. Four times 25 is 100. But, that's only 4 times 25. So, 16 times 25 is another 4 times. Four times 100 is 400."

Possible teacher response

"You said you broke apart the 16 into two factors and then multiplied one factor at a time. How does breaking up one of the numbers into factors help you solve the problem?"

Scenario 2: 16 x 26

Public Recording	Student's Explanation
10 + 6 = 16 10 x 25 = 250 6 x 25 = 150 250 + 150 = 400	"Well, I broke 16 into 10 + 6. Ten times 25 is 250 and 6 times 25 is 150. If you add the two together, you get 400."

Possible teacher response

"So, you broke apart the 16 into one ten and six ones. How did this help you to solve the problem? What do you know about multiplying two-digit numbers that helped you choose this strategy? Each strategy is different, yet each arrives at the same answer for 16 x 25. Why do you think this is so?"

Scaffold:

- When beginning Number Talks, make sure that the problems and quantities are accessible and within each child's zone of proximal development. The numbers must be accessible so that the students are solving the equations mentally.

- If you have students in your classroom who are performing at diverse instructional levels, select 3 different problems for students to solve at 3 different levels. Allow students to choose the problem which they will solve. Select problems with different size numbers so that all students have access to a problem and all students are working at a level that pushes them to their optimal level. For example:

  6 x 25   16 x 25   116 x 25

- As the students' flexibility, accuracy and efficiency improve, increase the rigor of the problems by adjusting the numbers or operations.

- Allow the students to document on paper their intermediate steps **as** they are solving the problem.

Test Prep:

Some children who understand many mathematical ideas do not fare well on a standardized test given in a multiple-choice format. Often, children guess a "letter" rather than reasoning through the problem. To improve children's test taking strategies while building number and operational sense, the following strategies are suggested:

- Pose a problem just as a problem would be posed with a "Number Talk." For example:

  There are 58 cases of soda in a warehouse. If there are 24 can of soda in each case, how many cans of soda are in the warehouse?

  Ask students to think about the problem in a way that makes sense to them.

- Only after the children have thought about the problem, show them an A., B., C., and D. response. Ask them to choose the answer that is closest to their thinking. For example:

  A. 1392
  B. 1362
  C. 1292
  D. 1262

- Ask students to publicly share the methods they used for solving the problem. For example:

  "I know that 24 is 10 + 10 + 4. 10 x 58 is 580. Another 10 cases would be 580 + 580 which is1160. 4 x 58 is 4 x 50 (which is 200) plus 4 x 8 (which is 32). 232 + 1160 is 1392. There are 1392 cans of soda in the warehouse. The answer is A."

  or

- *"I know that 24 is 20 + 4. 20 x 58 is 1160. 4 x 58 is 200 (4 x 50) plus 32 (4 x 8) which equals 232. 232 + 1160 is 1392. The answer is A."*

- When it fits the problem, facilitate conversations about the reasonableness of each choice (e.g. *"How did you know what operation to choose? Which choices could you have eliminated immediately? Why? Why would C not have been a reasonable choice? Why would D not have been a reasonable choice?*).

- As students "guess" less and "reason" more, pose the problem along with the A., B., C., and D responses. Ask students to think about the problem in a way that makes sense to them and then select the closest answer to their thinking. Ask students to share with a partner which responses they would eliminate immediately and why.

- The important piece is that students take the time to think and reason about the problem before they choose an answer (or guess). This must be a "habit of mind" whenever they are confronted with a problem to solve. Using this format once a week beginning very early in the school year could help students "break" the habit of guessing and assist in higher scores on standardized tests.

Notes about Number Talks:

A. Keep them short.

B. Encourage sharing and clarify students' thinking

C. Teach intentionally

- Start where your children are.

- Choose related sequences of problems.
- Focus students' thinking:

  See if you can . . .
  How many will there be if . . . ?
  What if . . . ?
  Can you use what you know about the last problem to help you think about this problem?

- Chart the students' thinking so that it can be saved and referred to later.

D. Create a safe and supportive environment

- Accept answers without praise or criticism.
- Allow students to ask questions of each other.
- Encourage students to listen to each other.
- Encourage students to self-correct.

E. Vary the Number Talks to meet the range of needs.

- Vary the sharing strategies used.

  Pair share
  Share whole group
  Explain someone else's strategy

- Vary the level of difficulty within a Number Talk.

  Use written problems
  Use story problems

- Record the students' thinking using correct notation on the board, on the overhead, or on chart paper.

F. Give students lots of practice with the same kinds of problems.

G. When planning or implementing a Number Talk, consider the following:

- How do students get their answers?
- Can students use what they know for related problems?
- How well can students verbalize their thinking?
- Are errors way off or are they reasonable?

H. The role of the teacher during a Number Talk is to facilitate and guide the conversation.

- The teacher purposefully chooses children to share strategies that will move the class toward computational fluency.
- The teacher asks questions that draw attention to the relationships among strategies.

It is important to focus on the mathematics, not just the variety of strategies. *Mathematically, why does the strategy work?*

Examples:

Multiplication

16 x 25	32 x 50	13 x 12
102 x 9	12 x 25	19 x 99

Division

125 ÷ 25	82 139 ÷ 9	205 ÷ 5
100 ÷ 15	143 ÷ 13	279 ÷ 9
54 ÷ 12	151 ÷ 11	16,000 ÷ 2,000

Inequalities

Greater than, less than, or equal to? 89 + 15 ☐ 85 + 19

Greater than, less than, or equal to? 89 x 15 ☐ 85 x 19

Greater than, less than, or equal to? 16 x 38 ☐ 18 x 36

Greater than, less than, or equal to? 32 x 18 ☐ 38 x 12

Expressions for students who need support

56 - 38	62 - 33	100 - 49
750 + 250	372 + 98	59 + 36
864 - 500	370 + 99	855-56
104 – 39	87 + 49	58 – 39
91 – 53	37 + 86	499 + 76
17 x 8	25 x 6	450 ÷ 45

20 x 4 x 2

15 x 30

16 x 5

Grade 4

Number Strings

Purpose:

- To use number relationships to solve problems and to learn number facts
- To use known facts and relationships to determine unknown facts
- To develop and test conjectures
- To make generalizations about mathematical relationships, operations and properties

Description:

This routine focuses on developing a sense of pattern and relationships among related problems. The task is at a higher level than merely recalling basic facts. Students identify and describe number patterns and relationships within and among equations. Students make conjectures about the patterns and relationships they notice. During this process, students explain their reasoning. Over time, students develop generalizations about important number relationships, operations and properties. These generalizations assist in solving problems and learning number facts.

Materials:

- Prepared list of number strings
- Whiteboard, chart paper, or overhead transparencies
- Student journals, student whiteboards, or scratch paper

Time: 15 minutes maximum per session

Directions:

Example:
a. 2 x 5 =
b. 4 x 5 =
c. 8 x 5 =
d. 16 x 5 =
e. 32 x 5 =
f. 48 x 50 =

1. Write equation "a" and ask students to solve mentally (e.g., 2 x 5 =). **Equation "a" should be easily accessible to all students.**
2. Have students check their answer with a partner.
3. Ask one student to share his/her solution with the class. Write the answer on the board to complete the equation (2 x 5 = 10).
4. Students show thumbs up or thumbs down for agreement or disagreement.
   - If there is agreement, go to equation "b."
   - If there is disagreement, facilitate a class conversation around the strategies the

student(s) used to arrive at the answer. Allow students to revise answers.

5. Give the students problem "b" to solve mentally (e.g., 4 x 5). Repeat, #'s 2, 3, and 4 above.

6. Write problem "c" (e.g., 8 x 5). Ask students how they could use what they know about the first two equations to solve this equation. Partner talk.

7. A volunteer shares his/her mathematical reasoning that derived an answer to this equation (e.g., "I know that the factor '2' in the first equation was multiplied by 2 to get the new factor '4' in the second equation. The '5' stayed the same so the product was also multiplied by 2: 10 x 2 = 20. Since the '8' in the third equation is 2 four times and the '5' stayed the same, then the product should also be multiplied four times: 10 x 4 = 40").

   Note: If students are having difficulty sharing relationships, ask questions such as the following:
   - *How are equations "a" and "b" alike?*
   - *How are equations "a" and "b" different?*
   - *Describe the relationship between the factors?*
   - *Describe the relationship between the products?*
   - *How can we use these relationships to predict the product for equation "c?"*

8. Write problem "d" (e.g., 16 x 5). Ask students to predict their answer to this problem. Students share their predictions with their partner and explain their thinking. Teacher writes predictions on the board.

9. A volunteer shares his/her mathematical reasoning that derived the answer to this equation (e.g., "16 x 5 =   " could lead to a discussion about quadrupling the "4 x 5 =   " equation or doubling the "8 x 5 =   " equation.)

10. Repeat steps 8 and 9 for equations "e," "f," and "g."

    Note: When students get to an equation that does not necessarily follow the same pattern (e.g., doubling), the discussion should yield many different strategies. (e.g., "48 x 5 =   " could be solved by adding the products of 16 x 5 and 32 x 5, or by multiplying the product of 8 x 5 by 6, or by multiplying the product of 2 x 5 by 24, etc.)

11. When the string is completed, facilitate a conversation about how relating a known equation can help students solve unknown equations. Listen for what relationships students notice throughout the string and how students are able to extend patterns beyond the string you have written. Ask students to make statements about the patterns and/or relationships that helped them to complete the string.

12. Examine the "conjectures" that the students share. Ask questions such as:
    - *Will doubling one factor always result in a doubled product? How can you prove your conjecture?*
    - *Will this always work? How can you prove your conjecture?*

Scaffold:

Begin with strings that grow in a predictable way and are easily accessible to all students

Possible Number Strings:

2 x 5 =	1 x 10 =	1 x 12 =	8 x 1 =
4 x 5 =	2 x 10 =	2 x 12 =	8 x 2 =
8 x 5 =	3 x 10 =	3 x 12 =	8 x 3 =
16 x 5 =	4 x 10 =	4 x 12 =	8 x 4 =
32 x 5 =	5 x 10 =	6 x 12 =	8 x 8 =
48 x 5 =	6 x 10 =	8 x 12 =	8 x 10 =
48 x 50 =	6 x 20 =	8 x 120 =	4 x 10 =
	6 x 200 =	8 x 121 =	12 x 10 =

12 ÷ 12 =	36 ÷ 3 =	8 ÷ 2 =	14 ÷ 7 =
12 ÷ 6 =	36 ÷ 6 =	16 ÷ 2 =	140 ÷ 7 =
12 ÷ 4 =	18 ÷ 6 =	32 ÷ 2 =	280 ÷ 7 =
12 ÷ 3 =	180 ÷ 6 =	48 ÷ 2 =	287 ÷ 7 =
12 ÷ 2 =	180 ÷ 12 =	48 ÷ 4 =	280 ÷ 14 =
12 ÷ 1 =	1800 ÷ 12 =	480 ÷ 4 =	2800 ÷ 14 =
12 ÷ 1/2 =	3600 ÷ 12 =	484 ÷ 4 =	2814 ÷ 14
12 ÷ 1/4 =		480 ÷ 40 =	

Generalizations to Develop Through These Strings:

Note: ***Do not tell students these generalizations. Ask students to make conjectures first and then ask them to test their conjectures using three or more examples. If the conjectures always holds true, then the students can make "generalizations."***

In multiplication, many strings begin by doubling one factor while leaving the other factor the same (e.g., 2 x 5 becomes 4 x 5). This always doubles the product accordingly (e.g., 2 x 5 = 10 becomes 4 x 5 = 20). The Big Idea associated with this pattern is: **By whatever amount the factor is multiplied, the product will be multiplied by the same amount.**

In division, this relationship holds true with the dividend and the quotient as well. **As the dividend is doubled** (8 ÷ 2 becomes 16 ÷ 2), **the quotient is doubled accordingly** (8 ÷ 2 = 4 becomes 16 ÷ 2 = 8).

The divisor has an inverse (opposite) relationship with the quotient. **As the divisor is multiplied by an amount, the quotient is divided by that same amount** (e.g., 36 ÷ 3 = 12 becomes 36 ÷ 6 = 6).

Sometimes the pattern is predictable because a factor is being doubled over and over, so the product doubles over and over, as well. But then, the pattern may change (e.g., 8 x 5 = <u>40</u>, 16 x 5 = <u>80</u>, 32 x 5 = <u>160</u>, then <u>48 x 5 = ?</u>).

In order to make sense of this situation a student must understand the associated Big Idea: **Numbers are the sum of more than one quantity** (e.g., 48 = 16 + 32). **The <u>Distributive Property</u> states that when a number is being multiplied by a particular factor, it is equivalent to multiplying the number by the parts that make up that factor** [e.g., 48 x 5 = (16 x 5) + (32 x 5)].

This Big Idea can help students develop an understanding of the relationships among numbers that will aid them in finding unknown products by relying on known facts (see *Using Strings to Learn Multiplication Facts* below).

For example:
Because 48 = 16 + 32, and students already know what 16 x 5 and 32 x 5 are, they can derive 48 x 5 as follows:

48 x 5 = (16 x 5) + (32 x 5)
<u>240</u> 80 + 160

The <u>Distributive Property</u> also states that when a dividend is being divided by a particular divisor (e.g., 2814 ÷ 14)**, it is equivalent to dividing the parts that make up that dividend by the same divisor and then adding the quotients**
[e.g., 2814 ÷ 14 = (2800 ÷ 14) + (14 ÷ 14)].

For Example:
Because 2814 = 2800 + 14, and students already know that 2800 ÷ 14 = 200 and 14 ÷ 14 = 1, they can derive 2814 ÷ 14 as follows:

2814 ÷ 14 = (2800 ÷ 14) + (14 ÷ 14)
<u>201</u> 200 + 1

<u>Using Strings to Learn Multiplication Facts</u>

Strings can be helpful to assist students to learn their multiplication facts as they learn to see the relationships among the facts.

<u>Example I</u>: If a student cannot remember 8 x 6, but knows 4 x 6, all the student has to do is double the product of 4 x 6 because 8 = 2 x 4.

4 x 6 = 24
8 x 6 = 48

<u>Example II</u>: If a student cannot remember 8 x 6, but knows 2 x 6 and 6 x 6, all the student has to do is find the product of these two equations and then find the sum of the products because 8 = 2 + 6.

2 x 6 = 12　　6 x 6 = 36　　8 x 6 = 48

Guiding Questions:

- What pattern(s) do you see?
- What stayed the same?
- What changed?
- How did it change?
- How did knowing the answers to the first equation help you figure out the answer to the next equation?
- Does this always work? How do you know?
- How are equations "a" and "b" alike?
- How are equations "a" and "b" different?
- What is the relationship between the factors?
- What is the relationship between the products?
- How can we use these relationships to predict the product for equation "c?"
- What is the relationship between the dividends?
- What is the relationship between the divisors?
- How can we use these relationships to predict the quotient for equation "c?"
- What is the relationship between the quotients?

Grade 4

Number of the Day

Purpose:

- To help develop students' flexibility with numbers and operations
- To develop understanding of number composition and part-whole relationships
- To explore equivalent arithmetic expressions

Description:

For "Number of the Day", students write equations that equal the number of days they have been in school. Students generate ways of combining numbers and operations to make that number.

Materials:

- Chart paper
- Individual white boards or journals

Time: 15 minutes maximum

Directions:

1. Post the chart paper.
2. Write the *Number of the Day* at the top of the chart paper.
3. Ask students to tell you everything they know about the number (e.g., 24; the number of sodas in four six packs; the number of eggs in two dozen; the number of crayons in a box of crayons; the number of classes at school; two tens and four ones; one cent less than a quarter).
4. Ask students to think of several models and equations that would represent the *Number of the Day.*
5. Ask students to represent the *Number of the Day* in at least four or more different ways.
6. Have students document these in their daily math journals.
7. Observe the students as they work and purposefully choose students to share whose representations will move the class towards further development of number and operational sense (e.g., 104 = 10 x 10 + 2 x 2).

8. Strategically call on those students who represented the number in meaningful ways that you would like to highlight.

9. Write those representations on the chart paper as students dictate.

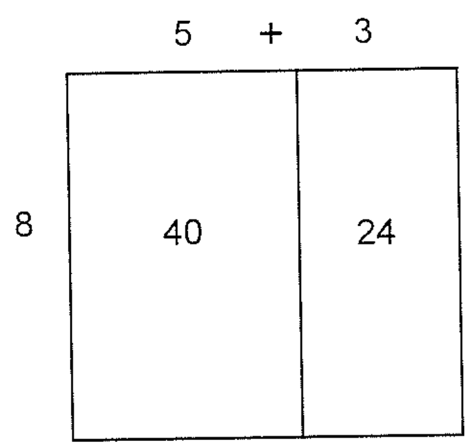
10. Finish by leading a class conversation around those representations that best connect to concepts recently learned. Be purposeful about the mathematics. Help students make mathematical connections whenever possible.

<u>Example:</u> The Number of the Day is "64"

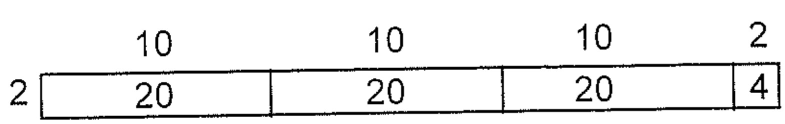
2 x 2 x 2 x 2 x 2 x 2 = 64	64 = 10 x 6 + 4
64 = 100 – 36	100 – 30 - 6 = 64
16 x 4 = 64	Five dozen and four more
50 + 10 + 4 = 64	LCIV is 64 in Roman Numeral

Rectangles showing an area model of 64 broken up in several ways:

8 x 8 = 64



2 x 32 = 64



Constraints

When students are familiar with the structure of *Number of the Day*, connect it to the number work they are doing in particular units. Add constraints to the equations to practice and reinforce different mathematical concepts. Ask students to include:

- Both addition and subtraction
- Three numbers
- Combinations of 10
- Multiplication
- Division
- Doubles
- Doubles plus one
- Multiples of 5 and 10
- Associative/Distributive/Commutative Properties of Addition/Multiplication
- Zero property
- Order property
- Draw a number line that places the number correctly
- Write equations with answer and equal sign on the left (45 = 15 + 20 + 10)
- Represent *number of the day* with manipulatives
- Represent *number of the day* with stories and pictures
- Represent *number of the day* with money
- Emphasize using tens
- Emphasize using hundreds

Grade 4

Number Lines

Purpose:

- To understand relationships between numbers
- To understand the relative magnitude of numbers.

Description:

Students place numbers on a number line. Students use what they know about one number to determine where a second number should be placed. As the types of numbers change and as the scale changes, students must use reasoning skills and their understanding of amounts and quantities to place the numbers.

Materials:

- A large, blank number line easily visible to all students during the routine time
- Attached blackline master of number lines

Time: 15 minutes maximum

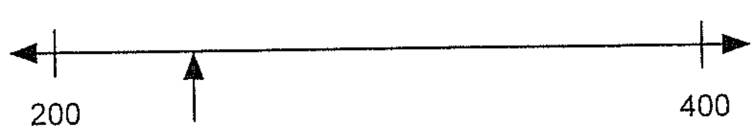
Caution: Always include arrows on both ends of your number line representations so students realize the number line is infinite; we are only looking at a section of the number line.

Directions:

VARIATION 1: ESTIMATION

1. Label 2 marks on the number line (e.g., 200 and 400).

2. Place an arrow somewhere between the 2 marks.



3. The class suggests reasonable values for the number at the arrow.
   The students should give reasons why the numbers they suggest are reasonable (e.g., "It looks like the arrow is about one fourth of the distance between 200 and 400. Since there are 200 numbers between 200 and 400, the arrow looks like it might be pointing to a number about 50 larger than 200, so I think it might be 250.").

Scaffold for Variation 1:

Give the students several numbers to choose from. Students select the number that makes the most sense to them and explain their reasoning. For example:

The arrow is pointing to which of the following numbers? Support your response with a mathematically convincing argument.

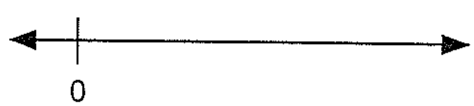
350, 250, 205, 380

Guiding questions for Variation 1:

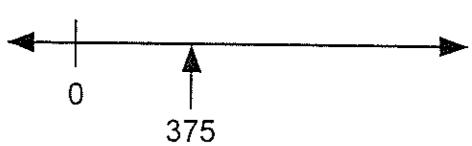
- Support your placement with a mathematically convincing argument.
- Name a number that is greater than this number.
- How much greater? Prove it on the number line.
- Name a number that is less than this number.
- How much less? Prove it on the number line.

<u>VARIATION 2: ESTIMATION</u>

1. Label the mark on the left with a zero.



2. Tell the students the arrow is pointing to a particular number (e.g., The arrow is pointing to 375).



3. Ask where other numbers would be. This helps students look at the relative positions of values. For example:
   About where would 750 be?
   About where would 190 be?
   About where would 300 be?
   Justify your answers with mathematically convincing arguments.

VARIATION 3: DECIMALS

1. Draw a number line with 11 marks, evenly spaced (this will give you ten intervals). Label the extremes "0" and "1."

2. Write decimals on index cards (one decimal number per card). Use decimals such as 0.46, 0.52, 0.7, 0.44, 0.8, 0.48, 0.32, 0.6, 0.08, etc.

3. Choose only 2 or 3 decimals to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line. Students must give a mathematically convincing argument as to why they are placing the number at this location.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Lead a conversation about any numbers that students believe might be misplaced. Choose a few numbers whose placement warrants further discussion (e.g., should .25 be placed to the left or to the right of .3? How do you know?)

9. Leave the numbers on the number line from one day to the next so that students can look at the decimals relative to other decimals they have worked with.

Scaffold for Variation 3:

- Use decimal numbers that go to the same decimal place (e.g., all tenths or all hundredths).

Extensions for Variation 3:

- Use labels other than 0 and 1 for the extremes (e.g., 3.1 and 4.1; 0.42 and 0.43).
- Include decimals that end in different places (e.g., 3, 0.3, 0.36).
- Use a number line with the intervals unmarked.

Guiding questions for Variation 3:

- Support your placement with a mathematically convincing argument.
- Name a decimal less/greater than yours.
- How do you know your decimal is less/greater than one half?
- Name another decimal equivalent to yours.

VARIATION 4: FRACTIONS

1. Label the extremes with "0" and "1."

2. Write different fractions on index cards (e.g., 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 8/8 as well as their equivalent fraction names).

3. Choose only 2 or 3 fractions to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give them time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line. Students must give a mathematically convincing argument as to why they are placing the number at this location.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Lead a conversation about any numbers that students believe might be misplaced. Choose a few numbers whose placement warrants further discussion (e.g., should 1/4 be placed to the left or to the right of 3/8? How do you know?

9. Leave the numbers on the number line from one day to the next so that students can look at the decimals relative to other decimals they have worked with.

Scaffold for Variation 4:

- Use accessible fractions (e.g., fourths and eighths or thirds and sixths).

Extensions for Variation 4:

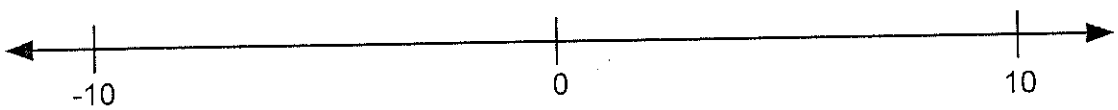
- Use a label other than 1 for the right-hand extreme (e.g., 2 or 3).
- Include mixed numbers and improper fractions in the numbers you place on the index cards.
- Integrate fractions that are not so "friendly" (e.g., fifths, sixths).

Guiding questions for Variation 4:

- Support your placement with a mathematically convincing argument.
- Name a fraction less/greater than yours. Prove it on the number line.
- How do you know your fraction is less/greater than one half?
- Name another fraction equivalent to yours.

VARIATION 5: INTEGERS

1. Label zero somewhere in the middle of the number line and label the extremes –10 and 10.



2. Write integers on index cards. Use numbers such as –1, -5, 3, 6, -8, 2, etc.

3. Choose only 2 or 3 integers to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Lead a conversation about any numbers that students believe might be misplaced. Choose a few numbers whose placement warrants further discussion (e.g., should -4 be placed to the left or to the right of 3? How do you know?)

9. Leave the numbers on the number line from one day to the next so that students can look at the amounts relative to other numbers they have worked with.

Extensions for Variation 6:

- Write integers on the outlying marks such as –3 and 5.
  Have the class decide the value for the mark in the middle.
  In this example the arrow is pointing to 1.

- Include numbers on the index cards that are less than the number marked on the left and are greater than the number marked on the right. For example, put –5 on a card. The students will place it an appropriate distance to the left of the –3 in the example. This will help students realize that each time we work with number lines, these number lines are just part of the infinite number line.
- Discuss relative distances between the numbers involved in the routine. (e.g., How far is it from –1 to 2?)

Guiding questions for Variation 6:

- Support your placement with a mathematically convincing argument.
- Name an integer that is greater than this number.
- How much greater? Prove it on the number line.
- Name an integer that is less than this number.
- How much less? Prove it on the number line.

VARIATION 7: VERY LARGE NUMBERS

1. Label the extreme to the left with a zero. Label the extreme to the right with1,000,000 or 2,000,000 or 3,000,000, etc. (a number in the millions).

2. Write large numbers (into the millions) on index cards.

3. Choose only 2 or 3 numbers to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Lead a conversation about any numbers that students believe might be misplaced. Choose a few numbers whose placement warrants further discussion (e.g., should 1,234,567 be placed to the left or to the right of 1,199,999? How do you know?

9. Leave the numbers on the number line from one day to the next so that students can look at the amounts relative to other numbers they have worked with.

Extensions for Variation 7:

- Write zero on the mark to the left. Vary the million you write on the mark to the right. Have the class decide on a reasonable value for the mark in the middle.

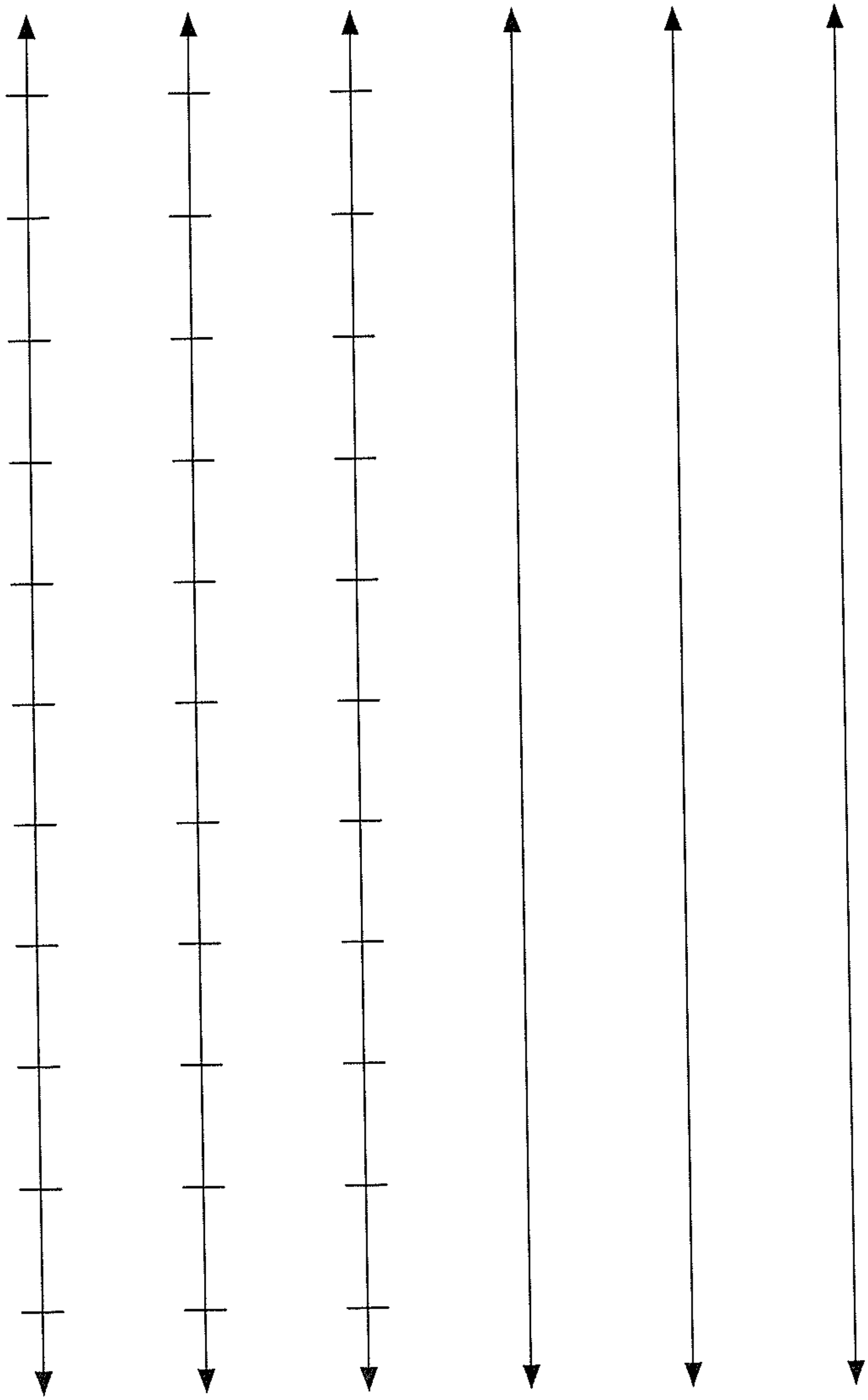
Change zero to a different value. This increases the rigor of labeling the unknown marks.

- Students place numbers that are relatively close together so that they must be more discriminating in their examination of their number (e.g., 42,721,000; 42,271,000; 42,172,000).
- Occasionally give students numbers that **don't** belong between the labeled marks (e.g., if you have labeled the outside marks "0" and "40,000,000, give students the number 53,531,671 to place. This reminds students that they are working with only **part** of the number line.

Guiding questions for Variation 7:

- Support your placement with a mathematically convincing argument.
- Name a number that is greater/less than this number.
- How much greater/less? Prove it on the number line.

- What number is a hundred more/less than your number?
- What number is a thousand more/less than your number?
- What number is ten thousand more/less than your number?



Grade 4

In and Out Boxes

Purpose:

Students recognize, describe, and generalize patterns and functional relationships. A function is a relationship in which two sets are linked by a rule that pairs each element of the first set with exactly one element of the second set.

Description:

Students analyze a set of number pairs to determine the rule that relates the numbers in each pair. The data are presented in the form of a function table (T-table) generated from an "In and Out Box." Students will describe rules for relating inputs and outputs and construct inverse operation rules.

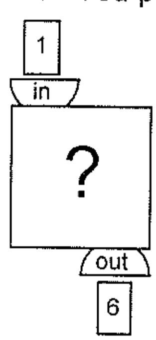
Materials:

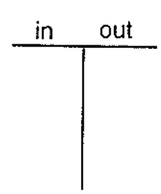
- T-table transparencies (What is the Rule?)
- In and Out Box transparency

Time: 15 minutes maximum

Directions:

1. Display the "In and Out box" on the overhead. Tell students that when you put a number in the box, a secret rule changes the number, and out comes a new number. Demonstrate with "1." You put "1" into the box, and "out" comes "6."





2. We organize the inputs and outputs with a T-table.

3. The job of the students is to determine the rule.

4. Write the first number in each column (e.g., "1" in the first column, "6" in the second column).

   Note: The example that we will be using may be complicated for early in the year. You may want to use a different "rule" when you introduce this routine to your class.

in	out
1	6

5. Ask pairs of students to discuss possible rules that could cause the number in the first column to become the number in the second column (e.g., add 5, multiply by 6, multiply by 3 and add 3).

6. Write the second number in each column (e.g., "2" in the first column, "9" in the second column).

in	out
1	6
2	9

7. Ask the students to check the rule they came up with for the first pair of numbers to see if the rule will apply to both sets of numbers. If the rule does not apply, ask students to think about a rule that could describe both sets of numbers.

8. Give students a third pair of numbers (e.g., 3 in the first column, 12 in the second). Does the rule still work? If not, students think about a different rule that could describe all three sets of numbers.

in	out
1	6
2	9
3	12

9. Ask students to write/discuss with their partner another pair of numbers that could fit their rule.

10. Ask for volunteers to share what they think other pairs of numbers could be that would fit the rule. Record these numbers on the transparency (without judgment as to whether they are correct or incorrect).

in	out
1	6
2	9
3	12
4	16
5	18
10	32
8	27

11. Partners discuss all the pairs of numbers on the transparency to see if they agree, or if there are inconsistencies in the pattern. Facilitate a class discussion about what operation(s) were working on each "in" to get each "out."

For Example:

For the teacher. DO NOT TELL

$1 + 1 + 1 + 3 = 6$
$3 \times 1 + 3 = 6$
$3(1 + 1) = 6$
$2 + 2 + 2 + 3 = 9$
$3 \times 2 + 3 = 9$
$3(2 + 1) = 9$
$3 + 3 + 3 + 3 = 12$
$3 \times 3 + 3 = 12$
$3(3 + 1) = 12$

in	out
1	6
2	9
3	12
~~4~~	~~16~~
5	18
~~10~~	~~32~~
8	27

Discuss similarities and differences, such as "Sue multiplied the first number by 3 and then added 3 more. But Julian added two 1's to the first number then multiplied **that** number by 3. Why do these two rules still work?"

12. Based on the above conversation for each of the specific numbers, have partners discuss what they believe the rule is for changing the first number in the pair to the second number in the pair. Students are now speaking in general terms, what must be done to any number, not just the numbers written on the table.

For the teacher. DO NOT TELL. In the example, each number in the "in" box is added to itself twice, plus three more; or, tripled, plus three more; or, multiplied by three, plus three more.

13. Ask for volunteers to share their rules. Record on the transparency the rules that students have generated. You will be recording the words, not numeric symbols (e.g., "The number in the 'out' box is equal to each number in the 'in' box multiplied by 3 plus three more").

14. Discuss how the rules are the same. Discuss how the rules are different. Do the rules accurately apply to each of the pairs of numbers? Facilitate a class discussion about how the rules work and how each is a different representation of the same pattern.

15. Once the class agrees on rules that will work, ask the partners to figure out other pairs of numbers that would follow the rules.

16. Ask students to volunteer some of their pairs as you record them. Have students explain why each pair of numbers follows the rule.

17. Ask partners to "translate" the rules written in words to the same rule written with mathematical symbols. For example: The rule may be to multiply the first number by 3 and then add 3; or, add 1 to the first number and then multiply that number by 3. In mathematical symbols, this would be $3n + 3$ and $3(n + 1)$. These expressions are 2 forms of the same rule, just written in different ways.

Note: Remind students that when we speak about "any number" instead of a specific number, we represent "any number" with a variable such as "n."

18. Give the students new "in" numbers. What would be the "out." How do you know?

Scaffold:

- Use smaller numbers and/or rules with only 1 operation (e.g., multiply by 2).

Extensions:

- Include rules with more than one step, as in the above example. (e.g., multiply by 2, then subtract 1).

- Use fractions or decimals as the numbers in the first column.

in	out
.25	.5
.5	1
1	2

in	out
1/2	1
1/4	1/2
1/8	1/4
1/16	1/8

- Give the students the first pair of numbers as explained above. Then, give the "out" number in the set. Ask students to find the "in" number in the set.

in	out
1	5
3	7
?	9
n	n + 4
n-4	n

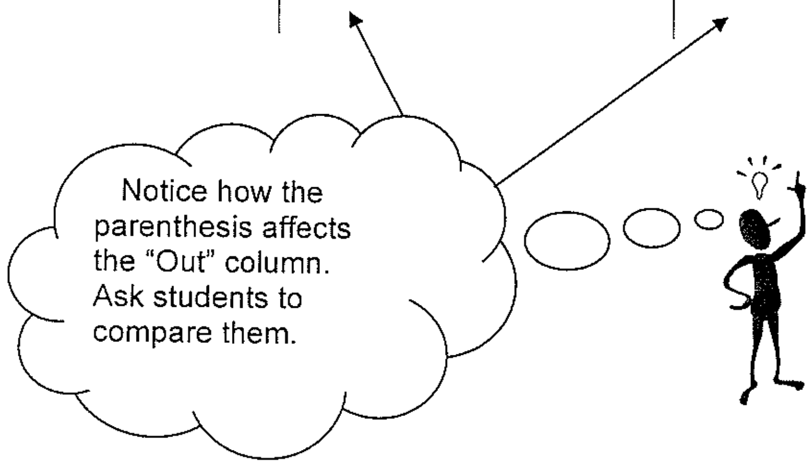
Note: Working backwards provides opportunities for using inverse operations to find missing elements and writing rules. For example: The In-Out rule is "n + 4;" the Out-In rule is "n - 4."

Examples:

In	Out
2	7
3	10
4	13
n	3n+1

In	Out
8	18
10	22
15	32
n	2(n+1)

In	Out
8	17
10	21
15	31
n	2n + 1



In	Out
1	1
2	2
4	4
9	9
n	n x n or n^2

In	Out
21	65
9	29
100	302
n	3n + 2

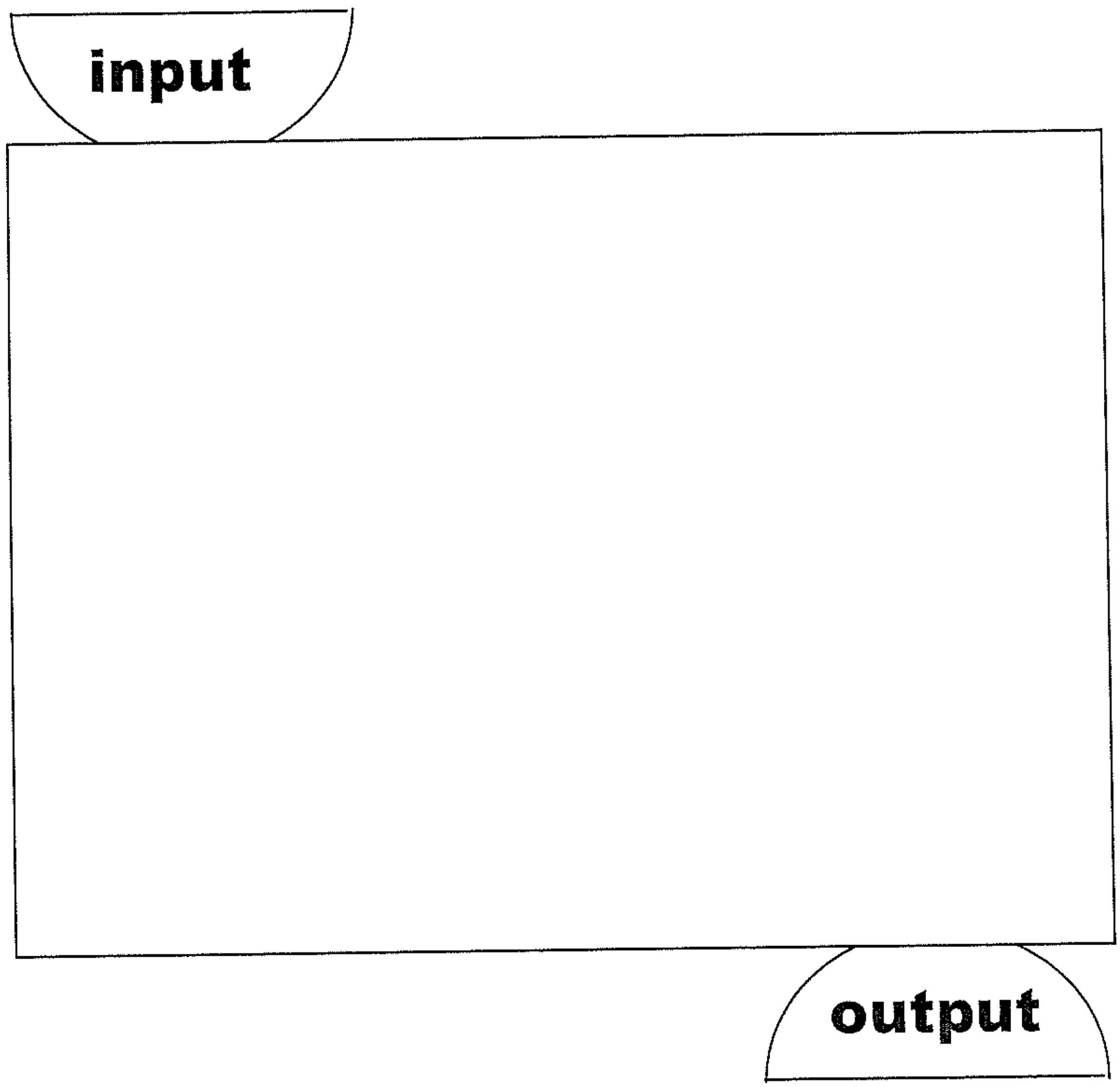
In	Out
1	3
2	6
3	9
n	3n

In	Out
1	2
2	3
3	4
n	n + 1

In	Out
1	0
2	1
3	2
n	n - 1

In	Out
1	1
2	3
3	5
n	2n - 1

In and Out Box



In	**Out**

What is the rule?

Concepts of Equality

Purpose:

- To develop an appropriate conception of equality and the equal sign
- To develop understanding that the equal sign denotes the relation between two equal quantities (rather than a command to carry out a calculation)

Description:

- Students are engaged in a discussion about the meaning of the equal sign.
- The context of this discussion is true/false and open number sentences.
- The number sentences provide a focus for students to articulate their ideas and to challenge their conceptions.
- The discussions assist in developing ways of thinking and communicating that embody the principles of algebraic reasoning.
- Students articulate mathematical principles that often are not explored or stated.
- Students must justify the principles that they propose in ways that convince others, and they must recognize and resolve conflicting assumptions and conclusions,

Materials:

Purposely planned number sentences and open number sentences. The numbers selected should be easily accessible to students. The focus is on the meaning of the equal sign, not on practice of operations.

Time:

15 minutes maximum

Pre-assessment

Before beginning this series of routines, ask your students to complete the following on a half-sheet of paper:

What number would you put in the box to make this a true number sentence?

$$8 + 4 = \square + 5$$

This information is for you. Do not discuss this problem with your students.

This problem was given to thirty typical elementary-grade classes. The responses were as follows:

	Response - Percent Responding			
	7	12	17	12 and 17
Grades 1 and 2	5%	58%	13%	8%
Grades 3 and 4	9%	49%	25%	10%
Grades 5 and 6	2%	76%	21%	2%

From: Falkner, Levi, & Carpenter, 1999

This data suggests that many elementary school students have serious misconceptions about the meaning of the equal sign as a relation between two equal quantities. Many seem to interpret the equal sign as a command to carry out a calculation (the answer is...).

This misconception limits students' ability to learn basic arithmetic ideas with understanding and their flexibility in representing and using those ideas. This creates even more serious problems as they move to algebra.

Directions:

1. Engage students in a general discussion about true/false number sentences or what it means for a number sentence to be true or false. Provide examples asking whether the number sentence is true or false and how they know it is true or false. For example:

   8 – 5 = 3   3 x 4 = 15   599 + 468 = 1,067

2. Once students are familiar with true/false number sentences, equations can be introduced that may encourage them to examine their conceptions of the meaning of the equal sign. Pose one equation at a time and lead a discussion as to whether the equation is true or false. Students must justify their claims. Do not tell. Lead a discussion and ask questions.

   For example:

4 + 5 = 9 4 + 5 = 4 + 5	9 = 4 + 5 4 + 5 = 5 + 4	9 = 9 4 + 5 = 6 + 3
3 x 4 = 12 3 x 4 = 3 x 4	12 = 3 x 4 3 x 4 = 4 x 3	12 = 12 3 x 4 = 2 x 6
15 – 7 = 8 15 – 7 = 15 – 7	8 = 15 – 7 15 – 7 = 7 - 15	8 = 8 15 – 7 = 16 – 8

24 ÷ 2 = 12	12 = 24 ÷ 2	12 = 12
24 ÷ 2 = 24 ÷ 2	24 ÷ 2 = 2 ÷ 24	24 ÷ 2 = 36 ÷3

Many of the examples above do not follow the familiar form with two numbers and an operation to the left of the equal sign and the answer to the right of the equal sign. This may throw some students into disequilibrium. Asking students to choose whether each number sentence is true or false can encourage them to examine their assumptions about the equal sign.

Note: We are trying to help students understand that the equal sign signifies a relation between two numbers. It is sometimes useful to use words that express that relation more directly (e.g., "Nine is the same as 4 plus 5").

3. Including zero in a number sentence may encourage students to accept a number sentence in which more than one number appears after the equal sign. For example:

   9 + 5 = 14 9 + 5 = 14 + 0 9 + 5 = 0 + 14 9 + 5 = 13 + 1

4. Open number sentences given after the corresponding true/false questions are a nice way to follow up on the ideas that came out of the discussion of the true/false number sentence. The question being asked is:

   "What number can you put in the box to make this number sentence true?"

4 + 5 = □	9 = 4 + □	9 = □
4 + 5 = □ + 5	4 + 5 = □ + 4	4 + 5 = □ + 3
□ = 4 + 5	4 + □ = 9	
3 x 4 = □	12 = 3 x □	12 = □
3 x 4 = □ x 4	3 x 4 = □ x 3	3 x 4 = □ x 6
□ = 3 x 4	3 x □ = 12	
15 – 7 = □	8 = 15 – □	8 = □
15 – 7 = □ – 7	15 – 7 = □ – 8	□ = 15 – 7
15 - □ = 8		
24 ÷ 2 = □	12 = 24 ÷ □	12 = □
24 ÷ 2 = □ ÷ 2	24 ÷ 2 = □ ÷ 3	□ = 24 ÷ 2
24 ÷ □ = 1		

Scaffolding:

The following are benchmarks to work toward as children's conception of the equal sign evolves.

1. Getting children to be specific about what they think the equal sign means (even if their thinking is incorrect). To do this, the conversation must go beyond just comparing the different answers to the problem. For example, in the problem 8 + 4 = □ + 5, some children might say:

   The equal sign must be preceded by two numbers joined by a plus or a minus and followed by the answer (resulting in an answer of 12 to this problem).

   You have to use all the numbers (resulting in an answer of 17 to this problem).

   Though this understanding is not correct, the articulation of conceptions represents progress.

2. Children accept as true some of the number sentences that are not of the form a + b = c (e.g., 8 = 5 + 3; 8 = 8; 3 + 5 = 8 + 0; or 3 + 5 = 3 + 5).
3. Children recognize that the equal sign represents a relation between two equal numbers (rather than "the answer is"). Children might compare the two sides of the equal sign by carrying out the calculation on each side.
4. Children are able to compare the mathematical expression without actually carrying out the calculation. For example: 8 + 4 = + 5

A child might say, "*I saw that the 5 over here (pointing to the 5 in the number sentence) was one more than the 4 over here (pointing to the 4 in the number sentence), so the number in the box had to be one less than the 8. So it's 7.*"

Guiding Questions:

- *Why do you think that?*
- *Why do you think that you cannot write number sentences that look like that?*
- *Do you agree or disagree with Student A? Why?*
- *Why do you believe this equation is true?*
- *Why do you believe this equation is false?*
- *What do you do when there is more than one number that follows the equal sign?*
- *How do you know that the number that you put in the box makes the number sentence true?*
- *How can you figure out the number that goes in the box without doing any calculation?*

Reference

Carpenter, Thomas P. Franke, Megan Loef, Levi, Linda, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School, Portsmouth, N.H.: Heinemann, 2003.
Falkner, Karen P., Levi, Linda, & Carpenter, Thomas P. 1999. "Children's Understanding of Equality: A Foundation for Algebra." *Teaching Children Mathematics 6*, 232-236.

Thinking Relationally

Purpose:

- To make the learning of arithmetic richer
- To think flexibly about mathematical operations
- To compare mathematical expressions without actually carrying out the calculation
- To help students recognize without having to calculate that the expressions on each side of the equal sign represent the same number
- To provide a foundation for smoothing the transition to algebra

  Note: In algebra, students must deal with expressions that involve adding, subtracting, multiplying, and dividing but that are not amenable to calculation (e.g., $3x + 7y - 4z$). They have to think about relations between expressions ($5x + 34 = 79 - 2x$) as they attempt to figure out how to transform equations in order to solve them.

Description:

Students are engaged in conversations about the relationships between numbers and how these relationships can be useful in finding solutions to problems. Students analyze expressions through the context of true/false and open number sentences. Students find ways to solve the problems by using number relations before calculating the answers.

Materials:

Purposely planned equations.

Note: Select equations that <u>cannot</u> be easily calculated. We want students to be motivated to look for relations. If equations can be easily calculated, the need does not exist to look for number relations.

Time:

15 minutes maximum

Directions:

1. To successfully implement relational thinking routines, the following classroom norms must be established:

   - Students explain their thinking
   - Students listen to one another
   - Alternative strategies for solving a given problem are valued and discussed
   - Solutions that involve more than simply calculating answers are not only accepted, but valued

2. You, the teacher, will need to make decisions based on the needs of your class. As you select problems:

   - Start with relatively easy problems and selected problems that provide an appropriate level of challenge based on what you have observed students doing on previous problems.
   - Select problems that will challenge students but not be too difficult for them.
   - Make decisions about what problems to use next based on students' responses to problems that they had already solved.

3. **Goal 1: For students to recognize that they do not always need to carry out calculations; they can compare expressions before they calculate.**

   Engage students in a general discussion about what it means for a number sentence to be true or false. Pose the following true/false problems (one at a time):

$$12 - 9 = 3$$
$$34 - 19 = 15$$
$$5 + 7 = 11$$
$$58 + 76 = 354$$

   Students explain how they know whether the number sentence is true or false. Students justify their solutions with their partner.
   Notice which students are calculating and which students are using relationships to determine whether the problems are true or false.

4. Pose the following true/false problem:

$$27 + 48 - 48 = 27$$

   Students justify their answers.
   This problem establishes that students do not necessarily have to calculate to decide if a number sentence is true or false.

5. Ask students to see if they can figure out in their heads whether the following problem is true or false (without adding or subtracting):

$$48 + 63 - 62 = 49$$

Students justify their answer.
This problem extends the idea that was used in the previous problem.

6. Pose the following true/false problem:

$$674 + 56 - 59 = 671$$

Students justify their solutions.

This problem is slightly more complicated than the preceding problem because students have to recognize that 59 breaks apart to 56 + 3 and that they can subtract 56 from the 56 given in the problem, and then they have to subtract 3 more from 674.

7. **Goal 2: To use properties of numbers and operations to think about relations between numerical expressions.**

Review open number sentences. Pose the following problem:

What number would you put in the box to make this a true number sentence?

$$7 + 6 = \square + 5$$

Students justify their solutions.

8. Pose the following problems (one at a time):

$$43 + 28 = \square + 42$$
$$28 + 32 = 27 + \square$$
$$67 + 83 = \square + 82$$

Students justify their solutions.
Children must recognize that they can use relational thinking to solve these problems without carrying out all the calculations.

9. Up until this point, boxes have been used to represent an unknown in an open number sentence. Students readily adapt to using letters to represent variables and unknowns. Pose the following problem:

$$12 + 9 = 10 + 8 + c$$

What is the value of c?

Students justify their solutions.
If students justify their answers with an explanation focusing on computation, ask how this problem could be solved without adding 12 + 9 or 10 + 8 (e.g., 10 is two less than 12 and eight is one less than nine, so c must be 3).

10. Pose a problem with larger numbers but the same general structure, as follows:

$$345 + 576 = 342 + 574 + d$$

What is the value of d?

Students justify their solutions.

11. Pose the following problem:

$$46 + 28 = 27 + 50 - p$$

What is the value of p?

Students justify their solutions.

12. When students have figured out how to deal with addition problems, move to subtraction problems. Pose the following problem:

$$86 - 28 = 86 - 29 - g$$

What is the value of g?

13. **Goal 3: Using relational thinking to learn multiplication facts.**

The following problems can be used to draw children's attention to relations among numbers that can make learning number facts easier.

- Knowing that addition and multiplication are commutative reduces the quantity of number facts that children have to learn by almost half.

True/False

6 x 7 = 7 x 6

What number would you put in the box to make this a true number sentence?

4 x 8 = 8 x □

- Understanding the relation between addition and multiplication makes it possible for students to relate the learning of multiplication facts to their knowledge of addition.

True/False

3 x 7 = 7 + 7 + 7
3 x 7 = 14 + 7
4 x 6 = 12 + 12
6 x 4 = 4 + 4 + 4 + 4 (false)

- Focusing on specific relationships among multiplication facts can make it possible for students to build on the facts they have learned.

True/False

3 x 8 = 2 x 8 + 8
6 x 7 = 5 x 7 + 7
8 x 6 = 8 x 5 + 6 (false)
7 x 6 = 7 x 5 + 7
9 x 7 = 10 x 7 - 7

Sample problems to assist in developing relational thinking

True/False (not all are true)

37 + 56 = 39 + 54

33 – 27 = 34 – 26

471 – 382 = 474 – 385

674 – 389 = 664 – 379

583 – 529 = 83 – 29

-4 = -8 + 4

37 x 54 = 38 x 53

60 x 48 = 6 x 480

5 x 84 = 10 x 42

64 ÷ 14 = 32 ÷ 28

42 ÷ 16 = 84 ÷ 32

-3 + 4 = -4 + 3

Sample problems for developing understanding of the properties of numbers and operations within numerical expressions

$73 + 56 = 71 + d$	$73 + 56 = 71 + 59 - d$
$68 + b = 57 + 69$	$68 + 58 = 57 + 69 - b$
$96 + 67 = 67 + p$	$96 + 67 = 67 + 93 + p$
$87 + 45 = y + 46$	$87 + 45 = 86 + 46 + t$
$92 - 57 = g - 56$	$92 - 57 = 94 - 56 + g$
$56 - 23 = f - 25$	$56 - 23 = 59 - 25 - s$
$74 - 37 = 75 - q$	$74 - 37 = 71 - 39 + q$
$-7 + 5 = n + 2$	$w - 5 = -2 - 4$
$3 + (-3) = 74 + v$	$-102 + (-2) = 98 + r$

Sample problems for developing base ten concepts

True/False (not all are true)

$56 = 50 + 6$	$47 + 38 = 40 + 7 + 30 + 8$
$87 = 7 + 80$	$24 + 78 = 78 + 20 + 2 + 2$
$93 = 9 + 30$	$63 - 28 = 60 - 20 - 3 - 8$
$94 = 80 + 14$	$63 - 28 = 60 - 20 + 3 - 8$
$94 = 70 + 24$	$0.78 = .078$
$246 = 24 \times 10 + 6$	$1.95 = 1.9500$

<u>Reference</u>

Carpenter, Thomas P. Franke, Megan Loef, Levi, Linda, <u>Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School</u>, Portsmouth, N.H.: Heinemann, 2003.